6) Local theory

Our goal is to describe the behavior of maps $f : X \circlearrowleft$ and their iterates locally at some fixed point $x_0 \in X$, i.e., on some neighborhood U of x_0 small enough. We need a few definitions

Def: Let X be a topological space, and $p \in X$ a point.
We say that two sets $A, B \subset X$ are equivalent at p: $A \sim_p B$, if there exists a open $U \subset X$ neighborhood of p so that $A \cap U = B \cap U \neq \emptyset$.
An equivalence class of $\sim_p$ is called a germ (of set) at p
We denote the germ of a set A at p as (A, p).
Analogously, let Y be another topological set.
~~We say that~~ Let $f : U_f \to Y$ and $g : U_g \to Y$ be two continuous function defined on neighborhoods U_f and U_g of p in X.
We say that $f \sim_p g$ if there exists a neighborhood V of p, $V \subset U_f \cap U_g$, so that $f|_V \equiv g|_V$. f_g can be seen as function from the germ (X, p) to Y.
An equivalence class of $\sim_p$ in $\mathcal{C}((X,p), Y)$ is called a germ (of continuous map) at p, denoted f_p.
Germ sets and germ of functions form a category.

Notice that germs of function are defined on germs of sets, and the value $f(p)$ is well defined.
If f_p is a germ of map at p, a representative is given by a map $f : U \to Y$ defined on a neighborhood U of p in X.

If $f_p : (X,p) \to Y$ is a germ of $\mathcal{C}^0$ map, $g_q : (Y, q) \to Z$ another at $q = f(p)$, then $h_p = g_q \circ f_p$ is well defined as a germ at p:
Take any representatives $f : U \to Y$ any $G : V \to Z$.

Then $f^{-1}(V)$ is an open neighborhood of p, and $h = g \circ f$ is well defined on $U \cap f^{-1}(V)$

If U', V' are other opens where f', g' are defined and give the same germ at p, then $h' = g' \circ f'$ is defined on $U' \cap f'^{-1}(V')$,

By definition, $\exists\, \tilde{U}, \tilde{V}$, $\tilde{U} \subset U \cap U'$, $\tilde{V} \subset V \cap V'$ so that $f|_{\tilde{U}} = f'|_{\tilde{U}}$, $g|_{\tilde{V}} \equiv g'|_{\tilde{V}}$, and $h \equiv h'$ on $\tilde{U} \cap f^{-1}(\tilde{V})$.

We say that a germ $f : (X,p) \to (Y,q)$ is invertible if there exist another germ $g : (Y,q) \to (X,p)$ so that $(g \circ f)_p = \mathrm{Id}_p$, $(f \circ g)_q = \mathrm{Id}_q$.

~~...~~ Similarly, two ~~sets~~ germs of sets (A,p) and (B,q) are homeomorphic if $\exists\, \phi : (A,p) \to (B,q)$ invertible germ.

We now focus on the case of holomorphic maps on Riemann surfaces.

In particular, we call a holomorphic germ the germ of a holomorphic map, defined on a neighborhood of a point $p \in X$ X Riemann surface.

By definition of Riemann surfaces, any germ (X,p) of a Riemann surface at a point p is biholomorphic to $(\mathbb{C},0)$, the biholomorphism given by a (germ of) chart centered at p.

Hence, holomorphic germs are (up to change of coordinates), always of the form $f : (\mathbb{C},0) \to \mathbb{C}$.

Being holomorphic maps analytical, f is determined by its expansion in Taylor series at 0.

Hence Holomorphic germs correspond to convergent ~~...~~ power series,

$$\mathcal{O}_{\mathbb{C},0} = \mathbb{C}\{z\} = \Big\{ \overbrace{\sum_{n\geq 0} a_n z^n}^{f(z)} \;\Big|\; a_n \in \mathbb{C},\ \exists\, r>0 \text{ so that } f(z) \text{ converges } \forall z, |z|<r \Big\}$$

Recall that the radius of convergence may be computed using the formula $\frac{1}{\rho} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

Hence $\sum a_n z^n$ is convergent $\Leftrightarrow$ $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < +\infty$.

$\Leftrightarrow$ ~~$\exists \alpha \in \mathbb{R}$~~ $\exists M > 0$ $\exists \alpha \in \mathbb{R}$ so that $|a_n| \leq M \cdot \alpha^n$ $\forall n \gg 0$. $(\Leftrightarrow \rho \leq \frac{1}{\alpha})$.

We denote by $\hat{\mathcal{O}}_{\mathbb{C},0} = \mathbb{C}[\![z]\!]$ the ring of formal power series
$= \{ \sum_{n\geq 0} a_n z^n \mid a_n \in \mathbb{C} \ \forall n \in \mathbb{N} \}$.

Notice that a holomorphic germ $f : (\mathbb{C},0) \to (\mathbb{C},0)$ is invertible
$\Leftrightarrow f'(0) = a_1 \neq 0$ (by implicit/inverse function theorem)

Similarly, for formal power series the value of f at 0 is well defined ($= a_0$), as the derivative $f' = \sum_{n\geq 1} n a_n z^{n-1}$, and hence $f^{(n)}(0) = n!\, a_n$.

A formal power series $f \in \mathbb{C}[\![z]\!]$ is a unit (i.e. invertible with respect to the product) $\overset{\text{def}}{\Leftrightarrow}$ $\exists g \in \mathbb{C}[\![z]\!]$ st $fg = 1 \Leftrightarrow a_0 \neq 0$.

One can define the composition $g \circ f$ of two formal power series as far as $f(0) = 0$ ($f \in z\mathbb{C}[\![z]\!]$).

Then $f \in z\mathbb{C}[\![z]\!]$ is invertible (with respect to composition) $\Leftrightarrow f'(0) = a_1 \neq 0$.

A fixed point p of a holomorphic selfmap $f : X \to X$ induces, by picking some coordinates at p, a holomorphic germ $f : (\mathbb{C},0) \circlearrowleft$.

Our aim is to find some coordinates "better" than others, where the Taylor series of f is as simple as possible.

This brings us to the following definition.

Def: let $f, \tilde{f} : (\mathbb{C},0) \circlearrowleft$ be two holomorphic germs fixing the origin (called also holomorphic fixed point germs)

We say that f and $\tilde{f}$ are holomorphically / topologically / formally (or analytically) conjugate if there exists a (invertible) holomorphic germ / C^0 germ / formal power series, $\Phi : (\mathbb{C},0) \to (\mathbb{C},0)$ so that $\Phi \circ f = \tilde{f} \circ \Phi$

denote $f \simeq \tilde{f}$, $\overset{hol}{\simeq}$, $\overset{top}{\simeq}$, $\overset{for}{\simeq}$

$$\begin{tikzcd} (\mathbb{C},0) \arrow[r, "\tilde{f}"] & (\mathbb{C},0) \\ (\mathbb{C},0) \arrow[u, "\simeq\ \Phi"] \arrow[r, "f"] & (\mathbb{C},0) \arrow[u, "\simeq\ \Phi"'] \end{tikzcd}$$

A (holomorphic / topological / formal) conjugacy invariant for a family $\mathcal{A} \subset \mathcal{O}_{(\mathbb{C},0)}$ is a function $I : \mathcal{A} \to S$ (S some set) so that if $f \simeq \tilde{f} \Rightarrow I(f) = I(\tilde{f})$. If the opposite holds ($f \simeq \tilde{f} \iff I(f) = I(\tilde{f})$), the invariant I is called complete.

Given a family $\mathcal{A}$ of (holomorphic) fixed point germs, a normal family $\mathcal{F}$ of $\mathcal{A}$ up to (hol./top/for) conjugacy is a family of germs so that $\forall f \in \mathcal{A}\ \exists \tilde{f} \in \mathcal{F}$, $f \simeq \tilde{f}$.

(Like this, we don't get much. we may ask that $\{\tilde{f} \in \mathcal{F} \mid f \simeq \tilde{f}\}$ is finite.

Notice that if $f \overset{hol}{\simeq} \tilde{f}$, then $f \overset{for}{\simeq} \tilde{f}$ and $f \overset{top}{\simeq} \tilde{f}$. The converse is not always true.

As an easy example of invariants:

Prop: The multiplier $f'(0)$ is a holomorphic and formal invariant of conjugacy.

The multiplicity $\mathrm{ord}_0(f)$ at zero is a holomorphic, formal and topological invariant of conjugacy

Proof. The first statement follows from the chain rule of the derivative: $\Phi\circ f = \tilde f\circ\Phi \Rightarrow \underbrace{\Phi'(0)}_{\neq 0} f'(0) = \tilde f'(0)\cdot\Phi'(0) \Rightarrow f'(0)=\tilde f'(0)$. (6.5)

For the second statement:

Lemma: $\forall f,g \in \mathbb{C}[[z]]$, $\mathrm{ord}_0(f)\geq 1 \Rightarrow \mathrm{ord}_0(g\circ f) = \mathrm{ord}_0(f)\cdot\mathrm{ord}_0(g)$.

Proof. Write $f(z) = a z^d(1+\overset{\delta(z)}{o(1)})$, $g(z) = b z^e(1+\overset{\varepsilon(z)}{o(1)})$, where $d\geq 1$, $e\geq 0$, $a,b\in\mathbb{C}^*$, and $o(1)$ denote suitable formal power series in $z\,\mathbb{C}[[z]]$.

Then $g\circ f = b\left(a z^d(1+\delta(z))\right)^e (1+\varepsilon(f(z))) = b a^e z^{de}(1+o(1))$.

Where we used the fact that $\mathrm{ord}_0(\phi)\geq 1 \Leftrightarrow \phi(0)=0$. $\Rightarrow \mathrm{ord}_0(g\circ f) = de$. $\square$

Then $\mathrm{ord}(\Phi\circ f) = 1\cdot\mathrm{ord}(f) = \mathrm{ord}(\tilde f)\cdot 1 = \mathrm{ord}(\tilde f\circ\Phi)$, and $\mathrm{ord}_0(f)$ is a formal, and hence holomorphic invariant.

Notice that if $f(z) = a z^d(1+\delta(z))$, we may find a determination of $\sqrt[d]{a(1+\delta(z))}$, i.e., a holomorphic function $\tilde a(1+\tilde\delta(z))$ so that $(\tilde a(1+\tilde\delta(z)))^d = a(1+\delta(z))$. By replacing z by $\tilde z = \tilde a z(1+\tilde\delta(z))$, (i.e. precomposing by $z\mapsto \tilde a^{-1} z(1+\tilde\delta(z))$) we may assume that $f(z) = z^d$ (not a conjugacy)

In particular any point $w\neq 0$ in a neighborhood of 0 has exactly d preimages. Being the number of preimages a topological invariant (homeomorphisms are bijections), $\mathrm{ord}_0(f)$ is a topological invariant. $\square$

We now study the classification of holomorphic selfmaps $f:(\mathbb{C},0)\circlearrowleft$ up to (holomorphic / formal / topological) conjugacy.

I Attracting case:

$f: (\mathbb{C},0) \circlearrowleft$, with multiplier $\lambda = f'(0)$ satisfying $0<|\lambda|<1$.

Theorem: Any attracting germ $f: (\mathbb{C},0)\circlearrowleft$ is holomorphically conjugate to $\tilde{f}(z) = \lambda z$. $(\lambda = f'(0))$.

The conjugacy Φ between f and $\tilde{f}$ is unique if we impose the value of $\Phi'(0)$.

Proof. $\exists$ we want to solve the equation $\Phi \circ f = \tilde{f} \circ \Phi$. $(*)$

Write $f(z) = \lambda z(1+\delta(z))$, $\tilde{f}(z) = \lambda z$, $\Phi(z) = z(1+\phi(z))$.

Then $(*)$ becomes: $\lambda z(1+\delta)(1+\phi\circ f) = \lambda z(1+\phi)$.

$\Leftrightarrow$ $(1+\delta)(1+\phi\circ f) = 1+\phi$.

Formally, a solution could be given by $(1+\phi) = (1+\delta)(1+\delta\circ f)\dots =$

$= \prod_{n\geq 1} (1+\delta\circ f^{n-1})$.

We want to show that this infinite product converges.

Lemma: $\prod_{n\geq 0}(1+\delta_n(z))^{a_n}$ converges on a neighborhood of $0 \Leftrightarrow$

$\sum a_n\delta_n(z)$ converges on a neighborhood of 0. $(a_n \in \mathbb{R}$, $\delta_n(0)=0)$

Proof (idea): Convergences are uniform on some disc $\{|z|\leq R\}$ for $R \ll 1$

Notice that if $\sum a_n\delta_n(z)$ converges then $|a_n\delta_n(z)| \to 0$ uniformly on z.

In this case we have $\dfrac{(1+\delta_n(z))^{a_n}-1}{a_n\delta_n} \xrightarrow[n\to\infty]{} 1$, and $\sum a_n\delta_n$ converges

$\Leftrightarrow \sum\left((1+\delta_n)^{a_n}-1\right)$ does. Hence we may suppose $a_n = 1$ $\forall n \in \mathbb{N}$.

$\prod(1+\delta_n(z))$ converges $\Leftrightarrow \log\prod(1+\delta_n(z)) = \sum\log(1+\delta_n(z))$ converges

Here $\log$ is the principal determination of logarithm, which is well defined because in order to have convergence, δ_n must be uniformly close to 0, hence away from -1.

But since $\dfrac{\log(1+\delta_n)}{\delta_n} \to 1$ when $\delta_n \to 0$, we have that

$\sum \log(1+\delta_n)$ converges $\Leftrightarrow$ $\sum \delta_n$ does $\square$

Hence we want to show that $\sum |\delta \circ f^{n-1}(z)| < \infty$ (uniformly on some neighborhood of the origin).

Since f is attracting ~~...~~ $\exists R \ll 1$, and Λ, $|\lambda| < \Lambda < 1$, so that

$|f(z)| \leq \Lambda |z| \quad \forall z \in D(0,R)$, and by induction $|f^n(z)| \leq \Lambda^n |z|$.

By continuity, $\exists C > 0$ so that $|\delta(z)| \leq C|z| \quad \forall z \in D(0,R)$.

Hence $\sum |\delta \circ f^{n-1}(z)| < C \cdot \sum_{n \geq 0} \Lambda^{n-1} |z| = \dfrac{C}{1-\Lambda}|z|$, and this product converges.

(!). Suppose there are two conjugacy maps Φ_1 and Φ_2.

Then $\Phi = \Phi_1 \circ \Phi_2^{-1}$ would be a local automorphism commuting with $\tilde{f}$.

Write $\Phi(z) = z \sum_{n \geq 0} \phi_n z^n$. Then: $\Phi \circ \tilde{f} = \tilde{f} \circ \Phi$ gives: (notice $\phi_0 = \Phi_1'(0)/\Phi_2'(0) = 1$)

$\lambda z \cdot \sum \phi_n (\lambda z)^n = \lambda z \sum \phi_n z^n$. From which we infer: $\lambda^n \phi_n = \phi_n \quad \forall n \in \mathbb{N}$.

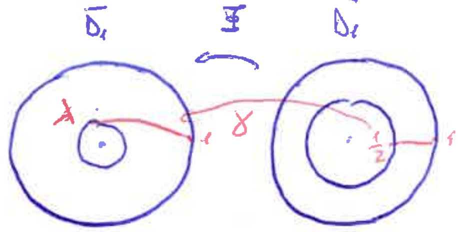
For $n = 0$, we get $\phi_0 = \phi_0$, a moot condition, but by hypothesis $\phi_0 = 1$.

For $n > 0$, since $|\lambda| < 1$, we get $\lambda^n \neq 1$ and $\phi_n = 0$. $\square$

In particular, for attracting germs, the multiplier is a complete invariant of formal and holomorphic conjugacy, and a normal family is given by linear maps $\{z \mapsto \lambda z\}$.

For the topological classification, we will show that:

Theorem: The ~~map~~ germ $f(z)=\lambda z$, $\lambda\in D(0,1)\setminus\{0\}$ is topologically conjugated to the map $f(z)=\frac{1}{2}z$.



Proof ~~[illegible]~~ Set $D_r=D(0,r)$, $C_r=\partial D_r$.

We will define a topologic conjugacy $\Phi:\overline{D}_1\to\overline{D}_1$, satisfying $\Phi\circ g=f\circ\Phi$, where $g(z)=\frac{1}{2}z$ and $f(z)=\lambda z$.

Set $A_0=\overline{D}_1\setminus D_{\frac{1}{2}}$ the closed annulus centered at 0 of radii $\frac{1}{2}$ and 1, and similarly $B_0=\overline{D}_1\setminus D_{|\lambda|}$.

Let $\gamma:[\frac{1}{2},1]\to B_0$ be a path joining $\gamma(\frac{1}{2})=\lambda$ and $\gamma(1)=1$, so that $|\gamma(t)|=2(1-|\lambda|)(t-1)+1$ (it suffices that $\#(\gamma([\frac{1}{2},1])\cap C_r)=1$ $\forall r\in[|\lambda|,1]$)

We define $\Phi_0:A_0\to B_0$ by the formula $\Phi_0(te^{i\theta})=\gamma(t)e^{i\theta}$ $\forall t\in[\frac{1}{2},1]$, $\theta\in[0,2\pi]$.

Notice that Φ_0 is a homeomorphism. Moreover, $\forall z\in C_1$, we have that

$$\Phi_0(g(z))=\Phi_0\left(\tfrac{1}{2}z\right)=\gamma\left(\tfrac{1}{2}\right)\cdot z=\lambda z=\lambda\Phi_0(z)\qquad (*)$$

Rem. The set $\overline{D}_1\setminus\overline{D}_{\frac{1}{2}}\underset{=:A_0'}{\subset} A_0$ is what is called a fundamental domain for the map $g|_{\mathbb{C}^*}$: $\forall z\in\mathbb{C}$, $\exists!\, z'\in A_0'$ such that $GO_g(z)=GO_g(z')$ (~~[illegible]~~ makes sense when g is invertible, as the action of $\{g^n\mid n\in\mathbb{Z}\}$)

↖ sometimes one asks for fundamental domains to be closed, and for the unicity to hold on a subset containing $\mathring{A}_0$.

Let now $z\in\mathbb{C}$ be any point. We define

$\Phi(z)=f^n\circ\Phi_0\circ g^{-n}(z)$ for any $n\in\mathbb{Z}$ so that $g^{-n}(z)\in A_0$, and $\Phi(0)=0$.

In other terms, $\Phi(z)=f^n(\Phi_0(g^{-n}(z)))$ $\forall z$, $\frac{1}{2^{n+1}}\leq|z|\leq\frac{1}{2^n}$.

By $(*)$, Φ is a well defined continuous map on $\mathbb{C}^*$, and it is also clearly continuous at 0. it is also bijective with continuous inverse defined 6.3

by $\Phi^{-1}(w) = \begin{cases} 0 & w=0 \\ g^n \circ \Phi_0^{-1} \circ f^{-n}(w) & \forall w,\ \frac{1}{|\lambda|^{n+1}} \leq |w| \leq \frac{1}{|\lambda|^n} \end{cases}$. □

Hence All attracting germs are conjugated to the same map $z \mapsto \frac{1}{2} z$.

Repelling case

Corollary: Any repelling germ $f: (\mathbb{C},0) \circlearrowleft$, with $f'(0) = \lambda$, $|\lambda| > 1$, is holomorphically and formally conjugated to $\tilde{f}(z) = \lambda z$ (with a unique conjugacy with prescribed derivative at 0); it is Topologically conjugated to $z \mapsto 2z$.

Proof: It follows directly from the analogous results for attracting germs applied to f^{-1}. □

Rem: These results work on any (algebraically closed) field, even in positive characteristic.

II Superattracting case

Theorem (Böttcher, 1904) Any superattracting germ $f: (\mathbb{C},0) \circlearrowleft$ is holomorphically conjugated to $\tilde{f}(z) = z^d$, $d \geq 2$.

The conjugacy map is unique up to multiplication by a $(d-1)$-th root of unity.

Proof: $\boxed{\exists}$ We write in some coordinates $f(z) = a z^d (1 + \delta(z))$, where $d \geq 2$ is the order at 0 of f, and $\delta: (\mathbb{C},0) \to (\mathbb{C},0)$ is a suitable holomorphic germ.

First of all, we may apply a linear change of coordinates $\Phi(z) = \alpha z$, $\alpha \neq 0$.

We get $\Phi \circ f \circ \Phi^{-1}(z) = \alpha f(\frac{1}{\alpha} z) = \alpha a \left(\frac{1}{\alpha}\right)^d z^d \left(1 + \delta\left(\frac{z}{\alpha}\right)\right)$

by taking α so that $\alpha^{d-1} = a$, we may assume $a = 1$.

The conjugacy equation then becomes: ($\Phi(z) = z(1+\phi(z))$)

$$\Phi \circ f = \tilde{f} \circ \Phi \iff z^d(1+\delta)(1+\phi\circ f) = z^d(1+\phi)^d \iff$$

$$(1+\delta)(1+\phi\circ f) = (1+\phi)^d.$$

A candidate solution is given by $(1+\phi(z)) = \prod_{n\geq 1}(1+\delta\circ f^{n-1}(z))^{\frac{1}{d^n}}$.

Notice that the expression makes sense as far as $\delta\circ f^{n-1}(z)$ is small.

But since f is contracting (actually superattracting), we have that $\forall \Lambda>0, (\Lambda<1)$ $\exists R>0$ s.t.

$|f(z)| \leq \Lambda|z|$ if $|z|\leq R$. By induction, $|f^n(z)| \leq \Lambda^n|z|$ $\forall z, |z|\leq R$.

Similarly, $\exists M>>0$ so that $|\delta(z)| \leq M|z|$.

By the criterium of convergence of infinite products, we need to check that $\sum_{n\geq 1}\frac{1}{d^n}\delta\circ f^{n-1}(z)$ is a convergent series. For $|z|\leq R$, we have:

$$\sum_{n\geq 1}\frac{1}{d^n}|\delta\circ f^{n-1}(z)| \leq \sum\frac{1}{d^n}\cdot M\cdot\Lambda^{n-1}|z| \leq \frac{M}{1-\Lambda}|z| < +\infty.$$

Hence $1+\phi$ converges, and $\Phi(z) = z(1+\phi(z))$ is the wanted conjugacy.

(!) If Φ_1 and Φ_2 are two conjugacies, then $\Phi = \Phi_1\circ\Phi_2^{-1}$ must be a local automorphism commuting with $\tilde{f}$.

Write $\Phi(z) = z\sum_{n\geq 0}\phi_n z^n$. The equation $\Phi\circ\tilde{f} = \tilde{f}\circ\Phi$ becomes:

$$\cancel{z^d}\sum\phi_n z^{nd} = \cancel{z^d}\left(\sum\phi_n z^n\right)^d$$

For $n=0$, we get $\phi_0 = \phi_0^d \Rightarrow \phi_0^{d-1} = 1$ and ϕ_0 is a d-root of unity.

We argue by contradiction, and assume that there is $n_0>0$ so that $\phi_{n_0}\neq 0$.

We take n_0 minimal

Then $\sum\phi_n z^{nd} = \phi_0 + \phi_{n_0}z^{n_0 d} + h.o.t.$ while

$$\left(\sum \phi_n z^n\right)^d = \phi_0^{d} + d\cdot \phi_{n_0} \phi_0^{d-1} z^{n_0} + h.o.t.$$

The coefficient $d\,\phi_{n_0}\phi_0^{d-1} \neq 0$, and the exponent $n_0 < n_0 d$, since $d \geq 2$.
This gives a contradiction (n_0 was minimal). $\square$

Rem: This result works for any field of characteristic 0, as far as $\forall a\ \exists \alpha$ such that $\alpha^{d-1} = a$. If not, $f = z \mapsto a z^d$ in any case.
In characteristic p the situation is far more complicated.
One can see it on the proof of uniqueness for example: $d\phi_{n_0}\phi_0^{d-1} = 0$ if $p \mid d$, $p = \operatorname{char} k$.
A more direct way to notice it is the example: $f(z) = z^p(1+z)$
Its derivative is $f'(z) = p z^{p-1} + (p+1) z^p = z^p$ (in char p.)
while $\tilde{f}(z) = z^p$ has everywhere vanishing derivative. Hence $f \not\sim \tilde{f}$ for.

Hence, in this case, formal, holomorphic and topological linearizations coincide (having $d = \operatorname{ord}_0(f)$ an invariant in all such cases)

$\underline{\text{III}}$ Parabolic / Tangent to the identity case.

We consider now the case of $f:(\mathbb{C},0)\circlearrowleft$ with $\lambda = f'(0)$ a root of unity: $\lambda = e^{2\pi i \frac{p}{q}}$, $q \in \mathbb{N}^*$, $p \in \mathbb{Z}$, $(p,q)=1$.

In local coordinates, we can write $f(z) = \lambda z(1+o(1))$.

Notice that the q-th iterate of f will be $f^q(z) = \underbrace{\lambda^q}_{1} z(1+o(1))$

Def: $f:(\mathbb{C},0)\circlearrowleft$ is called tangent to the identity if $\lambda = f'(0) = 1$.
In this case, we write $f(z) = z(1 + a z^r + o(z^r))$ for some $r \geq 1$, $a \neq 0$